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ALGEBRAIC STRUCTURES FOR $\oplus \Sigma_{n>1} L^2(Z/n)$ COMPATIBLE WITH THE FINITE FOURIER TRANSFORM

BY

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ABSTRACT. Let Z/n denote the integers mod n and let $\mathfrak{F}_n$ denote the finite Fourier transform on $L^2(Z/n)$. We let $\oplus\Sigma\mathfrak{F}_n = F$ operate on $\oplus\Sigma L^2(Z/n)$ and show that $\oplus\Sigma L^2(Z/n)$ can be given a graded algebra structure (with no zero divisors) such that $\mathfrak{F}(fg) = \mathfrak{F}(f)\mathfrak{F}(g)$. We do this by establishing a natural isomorphism with the algebra of theta functions with period i. In addition, we find all algebra structures on $\oplus\Sigma L^2(Z/n)$ satisfying the above condition.

Introduction. Let Z/n denote the integers modulo n and form

$$L = \oplus \sum_{n>1} L^2(Z/n).$$

Let $\mathfrak{F}_n$ denote the Fourier transform on $L^2(Z/n)$ and let $\mathfrak{F}$ denote the linear transformation of L such that

$$\mathfrak{F}|L^2(Z/n) = \mathfrak{F}_n$$

all n.

The main result of this paper may be stated as follows:

MAIN THEOREM. *L has, up to isomorphism, three structures as an algebra, L_α, $\alpha = 1, 2, 3$, such that*:

1. *if $f \in L^2(Z/n)$ and $g \in L^2(Z/m)$, $fg \in L^2(Z/n + m)$*;
2. *L_α, $\alpha = 1, 2, 3$, has no divisors of zero*;
3. for $f, g \in L_\alpha$, $\alpha = 1, 2, 3$,

$$\mathfrak{F}(fg) = \mathfrak{F}(f)\mathfrak{F}(g).$$

Further, by reordering L_α if necessary, we have

$$L_1 \approx \mathbf{C}[X_1, X_2^2, X_3^3]/(X_3^6 + X_2^6),$$

$$L_2 \approx \mathbf{C}[X_1, X_2^2, X_3^3]/(X_3^6 + X_1^4X_2^2 + X_2^6),$$

$$L_3 \approx \mathbf{C}[X_1, X_2^2, X_3^3]/(X_3^6 + X_1^4X_2^2),$$

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where ( ) *denotes the principal ideal of the term in the bracket and* $\mathbf{C}[\ ,\ ,\ ]$ *denotes the polynomial ring over the complex numbers* $\mathbf{C}$ *with the terms in the bracket as indeterminants.*

The uniqueness part of our main result is purely combinatorial and elementary in nature and contained in §1. However, the existence result rests on more technical considerations that require the following brief set of definitions.

Let N be the three dimensional Heisenberg group given by $(x, y, t) \in N$, with $x, y, t \in R$ and multiplication is given by

$$(x_1, y_1, t_1)(x_2, y_2, t_2) = \left(x_1 + x_2, y_1 + y_2, t_1 + t_2 + \tfrac{1}{2}(y_1x_2 - y_2x_1)\right).$$

Let Γ be the subgroup of N generated by the two elements (1, 0, 0) and (0, 1, 0). Then the homogeneous space $\Gamma\backslash N$ is a compact manifold we will call the Heisenberg manifold. One easily verifies that $J(x, y, t) = (-y, x, t)$ is an automorphism of N of order 4.

Let $L_{\mathbf{C}}$ denote the complexification of the Lie algebra of N. Then J induces an automorphism of $L_{\mathbf{C}}$, which we will also denote by J, and, up to scalar multiple, there exists a unique vector $X \in L_{\mathbf{C}}$ such that $J(X) = -\sqrt{-1}\,X$. Let

$$\theta = \{f \in C^\infty(N) | Xf = 0\}.$$

Now let

$$C_n(\Gamma) = \{f \in C^\infty(\Gamma\backslash N) | f(x, y, t + s) = e^{2\pi ins} f(x, y, t)\}$$

and let $\theta_n = C_n(\Gamma) \cap \theta$. It is well known (see [2]) that θ_n is empty for $n < 0$ and dim $\theta_n = n$ for $n > 0$. Further we will show that $J\theta_n = \theta_n$ and there exists an isometry

$$k_n \colon \theta_n \to L^2(Z/n)$$

such that $k_n(Jf) = \mathfrak{F}_n(k_n f)$ for $f \in \theta_n$. It is then obvious that $J(fg) = J(f)J(g), f, g \in \Sigma_{n>1}\theta_n$. Combining all this together will prove the existence of the algebras L_α, $\alpha = 1, 2, 3$.

One of the results of this paper is a formula for computing the multiplicities of the eigenvalues of $\mathfrak{F}_n$.

1. Uniqueness of algebraic structures. The main goal of this section will be the proof of the following result.

THEOREM 1. *Let* $L = \bigoplus\Sigma_{n>1} L^2(Z/n)$. *Let* L_A *be an algebra structure such that*:

1. *if* $f \in L^2(Z/n)$ *and* $g \in L^2(Z/m)$, *then* $fg \in L^2(Z/n + m)$;
2. $\mathfrak{F}(fg) = \mathfrak{F}(f)\mathfrak{F}(g)$;
3. L_A *has no divisors of zero.*

If L_A exists, it is of the form

$$L_A = \mathbf{C}[X_1, X_2^2, X_3^3]/\left(X_3^6 + aX_1^4X_2^2 + bX_2^6\right)$$

where a and b are both not zero. Further the images of the monomials $X_1^{n_1}X_2^{2n_2}X_3^{3n_3}$ in L_A are eigenvectors with eigenvalue $(-1)^{n_2}(i)^{n_3}$.

Proof. Let $\mathfrak{F}_n$ be the Fourier transform on $L^2(Z/n)$. Then $\mathfrak{F}_n$ has eigenvalues ± 1, $\pm i$ for $n \geqslant 5$ and the eigenvectors span $L^2(Z/n)$ for all n. We will use these basic facts to study L_A.

By a direct computation we know that there are elements $X_j^j \in L^2(Z/jZ)$, $j = 1, 2, 3$, such that

$$\mathfrak{F}_j(X_j^j) = \begin{cases} 1, & j = 1, \\ -1, & j = 2, \\ i, & j = 3. \end{cases}$$

The multiplicity of $\mathfrak{F}$ immediately implies Table A for $n = 1, 2, 3$. For $n = 4$, it follows from Condition 3 that X_1^4, $X_1^2\, X_2^2$, $X_1X_3^3$ are linearly independent. To prove that Table A holds for $n = 4$ we must show X_1^4 and X_2^4 are linearly independent. However if $aX_1^4 + bX_2^4 = 0$, a, b not both zero, Condition 3 would be contradicted. The cases $n = 5, 6$ follow in exactly the same way. The case $n = 6$ is special for t is the first time a relation occurs between X_1, X_2^2, X_3^3. Since X_3^6 does not occur in Table A for $n = 6$ it must be linearly dependent on $X_1^4X_2^2$, X_2^6. Hence $X_3^6 = aX_1^4X_2^2 + bX_2^6$, a, b not both zero. We shall show that this is the only relation which can occur.

Consider the n-monomials in $L^2(Z/nZ)$, $n \geqslant 7$,

$$\begin{aligned} &X_1^{n-j}X_2^j, && j \equiv 0 \bmod 2, \quad 0 \leqslant j \leqslant n, \\ &X_1^{n-k}X_2^{k-3}X_3^3, && k \equiv 1 \bmod 2, \quad 0 < 3 \leqslant k \leqslant n. \end{aligned} \qquad *$$

The monomials of $*$ coincide with those of Table A and Condition 2 immediately implies that they are eigenvectors with the appropriate eigenvalues with respect to $\mathfrak{F}_n$. We claim they are linearly independent. The proof is by induction. Assume n is even. The case n odd follows in exactly the same way. The monomials of $*$ all contain a positive power of X_1 except for X_2^n. Since by induction the $(n - 1)$-monomials

$$\begin{aligned} &X_1^{(n-1)-j}X_2^j, && j \equiv 0 \bmod 2, \quad 0 \leqslant j \leqslant n-1, \\ &X_1^{(n-1)-k}X_2^{k-3}X_3^3, && k \equiv 1 \bmod 2, \quad 0 < 3 \leqslant k \leqslant n-1, \end{aligned}$$

are linearly independent, Condition 3 implies the monomials of $*$, excluding X_2^n, are linearly independent. If $n \equiv 0 \bmod 4$ then $\mathfrak{F}(X_2^n) = X_2^n$. If $*$ is linearly dependent then we have a nontrivial relation

$$a_1X_1^n + a_2X_1^{n-4}X_2^4 + \cdots + X_2^n = 0.$$

Some $a_j \neq 0$, otherwise $X_2^n = 0$, contradicting Condition 3. Thus, writing $\xi = X_1^4/X_2^4$, we can write $X_2^n G(\xi) = 0$ where $G(0) = 1$. But we can then factor $G(\xi)$ and obtain $(\tilde{a}X_1^4 + \tilde{b}X_2^4)W = 0$, which contradicts Condition 3. The case $n \equiv 2 \bmod 4$ follows in exactly the same way.

To see that $X_3^6 + aX_1^4X_2^2 + bX_2^6 = 0$ is the only relation which can occur, we simply note that any relation between nth-degree monomials can modulo $X_3^6 + aX_1^4X_2^2 + bX_2^6$ be replaced by a relation involving only the monomials of $*$. The theorem is proved.

TABLE A

Eigenvector	Eigenvalue	Multiplicity
$X_1^{n-4k}X_2^{4k}, 0 \leqslant k \leqslant [\frac{n}{4}]$	1	$[\frac{n}{4}]+1$
$X_1^{n-(4k+1)}X_2^{4k-2}X_3^3, 1 \leqslant k \leqslant [\frac{n-1}{4}]$	$-i$	$[\frac{n-1}{4}]$
$X_1^{n-(4k+2)}X_2^{4k+2}, 0 \leqslant k \leqslant [\frac{n-2}{4}]$	-1	$[\frac{n-2}{4}]+1$
$X_1^{n-(4k+3)}X_2^{4k}X_3^3, 0 \leqslant k \leqslant [\frac{n-3}{4}]$	i	$[\frac{n-3}{4}]+1$

THEOREM 2. *Using the notation of Theorem* 1, *assume that any of the algebras* L_A *exist. Then all the algebras* L_A *exist.*

PROOF. Since $\mathfrak{F}_n$ is of order 4 it is completely reducible and so uniquely determined by its eigenvectors and eigenvalues. Now $X_1^{n_1}X_2^{2n_2}$, $n_1 + 2n_2 = n$, and $X_1^{n_1}X_2^{2n_2}X_3^3$, $n_1 + 2n_2 + 3 = n$, span the vector space of dimension n invariant under $\mathfrak{F}_n$ and depend only on the additive structure of L_A. Since L_A has the same representative monomials for all algebraic structures we have, if

$$\mathfrak{F}_n(X_1^{n_1}X_2^{2n_2}X_3^3) = i(-1)^{n_2}X_1^{n_1}X_2^{2n_2}X_3^3, \qquad n_1 + 2n_2 + 3 = n,$$

and

$$\mathfrak{F}_n(X_1^{n_1}X_2^{2n_2}) = (-1)^{n_2}X_1^{n_1}X_2^{2n_2}, \qquad n_1 + 2n_2 = n,$$

then $\mathfrak{F}_n$ is the Fourier transform on $L^2(Z/n)$ if L_A exists for some algebraic structure.

Next observe that as L_A is generated by X_1, X_2^2 and X_3^3 we have only to verify that the following three equations hold to prove our theorem:

$$\mathfrak{F}(X_1 \cdot Y) = \mathfrak{F}(X_1)\mathfrak{F}(Y), \quad \mathfrak{F}(X_2^2 \cdot Y) = \mathfrak{F}(X_2^2)\mathfrak{F}(Y),$$
$$\mathfrak{F}(X_3^3 \cdot Y) = \mathfrak{F}(X_3^3)\mathfrak{F}(Y),$$

where Y is a monomial. By our tables in the proof of Theorem 1 we need only verify the third relation. For Y of the form $X_1^{n_1}X_2^{2n_2}$ this again is obvious and so it remains only to verify by direct computation that

$$\mathscr{F}(X_3^3 \cdot X_1^{n_1}X_2^{2n_2}X_3^3) = \mathscr{F}(X_3^3)\mathscr{F}(X_1^{n_1}X_2^{2n_2}X_3^3)$$

to complete the proof of Theorem 2.

Let us now turn our attention to which of the algebras L_A are isomorphic. It is clear that if both a and b are not zero all the algebras L_A are isomorphic. Similarly if $a = 0$ and $b \neq 0$, all the algebras L_A are isomorphic as is the case when $b = 0$ and $a \neq 0$. Thus we have easily

THEOREM 3. *Up to isomorphism there are at most three algebras that satisfy Theorem* 1. *They are* $\mathbf{C}[X_1, X_2^2, X_3^3]/(\mathbf{Q})$ *where* $\mathbf{Q} = X_3^6 + X_1^4X_2^4$ *or* $X_3^6 + X_2^6$ *or* $X_3^6 + X_1^4X_2^2 + X_2^6$.

THEOREM 4. *Each of the principal ideals* $(X_3^6 + X_1^4X_2^2)$, $(X_3^6 + X_2^6)$ *and* $(X_3^6 + X_1^4X_2^2 + X_2^6)$ *are prime in* $\mathbf{C}[X_1, X_2^2, X_3^3]$.

PROOF. By standard results a principal ideal in $\mathbf{C}[X_1, X_2^2, X_3^3]$ is prime if and only if it is irreducible in this ring. Now

$$X_3^6 + X_2^6 = X_2^6(\xi^6 + 1)$$

where $\xi = X_3/X_2$. Hence any factoring of $X_3^6 + X_2^6$ has factors of the same degree in X_3 and X_2. But this is impossible in $\mathbf{C}[X_2^2, X_3^3]$.

Similarly in $\mathbf{C}[X_1, X_2, X_3]$,

$$X_3^6 + X_1^4X_2^2 = (X_3^3 + iX_1^2X_2)(X_3^3 - iX_1^2X_2)$$

where each term is irreducible. Hence $X_3^6 + X_1^4X_2^2$ is irreducible in $\mathbf{C}[X_1, X_2^2, X_3^3]$.

To show that $X_3^6 + X_1^4X_2^2 + X_2^6$ is irreducible, consider

$$(X_3^3 + P(X_1, X_2))(X_3^3 + Q(X_1, X_2)) = X_3^6 + X_1^4X_2^2 + X_2^6.$$

This implies that

$$-P^2 = X_1^4X_2^2 + X_2^6 = X_2^2(X_1^2 + iX_2^2)(X_1^2 - iX_2^2).$$

But this is impossible and $X_3^6 + X_1^4X_2^2 + X_2^6$ is irreducible.

We may summarize our results so far by the following statement: If one L_A exists, we have proven our main theorem. The proof of the existence of this L_A will be the goal of the rest of this paper.

2. Finite Heisenberg groups. In this section we give a brief discussion, without proof, of certain finite nilpotent groups that we will call the finite Heisenberg groups. Perhaps the simplest definitions of the finite Heisenberg groups are given as follows: The group Γ as defined in the Introduction has a faithful matrix representation

$$\begin{bmatrix} 1 & n_1 & n_3 \\ 0 & 1 & n_2 \\ 0 & 0 & 1 \end{bmatrix}$$

where n_i, $i = 1, 2, 3$, are in the integers Z. We define the n-Heisenberg group, $\Gamma/(n)$, by reducing each of the above integer matrices modulo n. Hence $\Gamma/(n)$ is the matrix group over the ring Z/n of the form

$$\begin{bmatrix} 1 & a_1 & a_3 \\ 0 & 1 & a_2 \\ 0 & 0 & 1 \end{bmatrix}, \qquad a_i \in Z/n, i = 1, 2, 3.$$

It follows easily that $\Gamma/(n)$ satisfies an exact sequence

$$1 \to Z/n \xrightarrow{p} \Gamma/(n) \xrightarrow{h} Z/n \oplus Z/n \to 1$$

with $p(Z/n)$ as the center of $\Gamma/(n)$, $z(\Gamma/(n))$. We now give the description of $\Gamma/(n)$ in terms of group extensions. Let $p(Z/n)$ be identified with $\exp(2\pi i a_3/n)$, $0 \leq a_3 < n$, and let $(a_1, a_2) \in Z/n \oplus Z/n$. Let $\gamma_1, \gamma_2 \in \Gamma/(n)$ be such that

$$h(\gamma_1) = (1, 0), \quad h(\gamma_2) = (0, 1).$$

Then the group extension $\Gamma/(n)$ is then completely given by $[\gamma_1^{a_1}, \gamma_2^{a_2}]$ where the symbol [,] denotes the commutator of the elements in the bracket. An elementary computation then yields that $[\gamma_1^{a_1}, \gamma_2^{a_2}] = \exp 2\pi i(a_1 a_2/n)$. We thus see that $\Gamma/(n)$ is built from the pairing of Z/n and its character group to the circle group given by the duality of finite abelian groups.

Let us now review the basic facts about the unitary representation theory of the groups $\Gamma/(n)$ as presented for instance in Auslander and Brezin [1]. Let A be a maximal abelian subgroup of $\Gamma/(n)$. Clearly A contains $z(\Gamma/(n))$. Let χ be a character which is nontrivial when restricted to $z(\Gamma/(n))$. Then inducing χ from A to $\Gamma/(n)$ gives an irreducible unitary representation, $\text{Ind}(\chi, A)$, of $\Gamma/(n)$. Further, any irreducible unitary representation which when restricted to $z(\Gamma/(n))$ is nontrivial and equal to χ restricted to $z(\Gamma/(n))$ is unitarily equivalent to $\text{Ind}(\chi, A)$. Since A is a direct sum of two copies of $z(\Gamma/(n))$ any character on $z(\Gamma/(n))$ can be extended to A. Thus all irreducible unitary representations of $\Gamma/(n)$ are of the form $\text{Ind}(\chi, A)$.

Now let $\Gamma/(n) = B' \times A$ where $B' \subset \Gamma/(n)$ and B' is isomorphic to $Z/(n)$. Then by the standard theory of induced representation, $\text{Ind}(\chi, A)$ acts on the vector space $L^2(B')$ or $L^2(Z/(n))$. Now let B be the maximal abelian subgroup of $\Gamma/(n)$ which contains B'. We may similarly form $\text{Ind}(\mu, B)$ where μ is a character on B. Now if $\mu|z(\Gamma/(n)) = \chi|z(\Gamma/n)$ we know from the above discussion that $\text{Ind}(\mu, B)$ and $\text{Ind}(\chi, A)$ are unitarily equivalent. Hence it is important to know that an intertwining operator is between these two representations. The answer is given by the following result that we will state without proof.

THEOREM 5. *Let A and B be the maximal abelian subgroups of $\Gamma/(n)$ defined* *above and let* χ *and* μ *be characters on* A *and* B, *respectively, such that* $\chi | z(\Gamma/(n)) = \mu | z(\Gamma/(n))$ *and is nontrivial. Then viewing* $L^2(\Gamma/(n)/A)$ *and* $L^2(\Gamma/(n)/B)$ *as* $L^2(Z/n)$ *we have*

$$\mathfrak{F}_n^{-1}\operatorname{Ind}(\chi, A)\mathfrak{F}_n = \operatorname{Ind}(\mu, A).$$

3. The left action on $C_n(\Gamma)$ and the spaces θ_n. We begin this section by reviewing, again without proof, the basic facts about the left action on $C_n(\Gamma)$ as defined by Auslander and Brezin [**1**].

Let $C_n(\Gamma) \subset C^\infty(\Gamma \backslash N)$ be as defined in the Introduction. Define $\Gamma(n)$ as the subgroup of N generated by Γ and the element $(0, 0, 1/n) \in z(N)$, where $z(N)$ denotes the center of N. Now let $f \in C_n(\Gamma)$. Then we may view f as a function on N such that $f(\gamma g) = f(g)$, $\gamma \in \Gamma$ and $g \in N$. Now

$$\begin{aligned} f((0,0,1/n)(x,y,t)) &= f(x,y,t+1/n) \\ &= \exp 2\pi i n(1/n) f(x,y,t) = f(x,y,t). \end{aligned}$$

Hence if $f \in C_n(\Gamma)$ then $f \in C^\infty(\Gamma(n) \backslash N)$. Now let $\Lambda(n)$ be the subgroup of N with generators $(1/n, 0, 0)$, $(0, 1/n, 0)$, $(0, 0, 1/n^2)$. Note that $\Gamma \subset \Gamma(n) \subset \Lambda(n)$ and that $\Gamma(n)$ is normal in $\Lambda(n)$. It is an elementary computation to verify that $\Lambda(n)/\Gamma(n)$ is isomorphic to $\Gamma/(n)$. On $C_n(\Gamma)$ we define two representations, one of N and the other of $\Lambda(n)/\Gamma(n)$ by

$$\begin{aligned} (R(g)f)(\Gamma(n)h) &= F(\Gamma(n)hg), \qquad h, g \in N, f \in C_n(\Gamma), \\ (L(\lambda)f)(\Gamma(n)h) &= F\big(\Gamma(n)\lambda^{-1}h\big), \quad \lambda \in \Lambda(n). \end{aligned}$$

Because $\Gamma(n)$ is in the kernel of the representation L, we will view L as a representation of $\Gamma/(n)$ or $\Lambda(n)/\Gamma(n)$. We will call this the left action of the n-Heisenberg group on $C_n(\Gamma)$. Further, it is clear that

$$L(\lambda)R(g) = R(g)L(\lambda), \qquad g \in N, \lambda \in \Lambda(n).$$

We will now review some of the facts from Auslander and Tolimieri [**2**, Chapter II], but from a slightly more intrinsic point of view than presented there. We observe that the one parameter subgroups of N are exactly the one dimensional linear subspaces in the (x, y, t) coordinate system. One verifies immediately that J: $(x, y, t) \to (-y, x, t)$ is an automorphism of N such that $J(\Gamma) = \Gamma$. Hence J induces a diffeomorphism of $\Gamma \backslash N$ onto itself and a linear mapping of $C_n(\Gamma)$ onto itself. Further if $f \in C_n(\Gamma)$ and $g \in C_m(\Gamma)$, then $fg \in C_{n+m}(\Gamma)$ and $J(fg) = J(f)J(g)$.

Now let $L_{\mathbf{C}}$ denote the complexification of the Lie algebra of N and let the Lie algebra L of N be viewed as left invariant vector fields on N. Now J induces an automorphism of $L_{\mathbf{C}}$, also denoted by J. We find that a basis for $L_{\mathbf{C}}$ is given in the (x, y, t) coordinate system by the left invariant vector fields

$$X = \frac{\partial}{\partial x} + \frac{1}{2} y \frac{\partial}{\partial t}, \quad Y = \frac{\partial}{\partial y} - \frac{1}{2} x \frac{\partial}{\partial t}, \quad Z = \frac{\partial}{\partial t}.$$

Now consider V_{-i} the eigenvalue $-\sqrt{-1}$ space under J acting on $L_{\mathbf{C}}$. Then V_{-i} has a basis

$$X + iY = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) - \frac{1}{2}(x + iy)\frac{\partial}{\partial t}$$

or the Lewy operator. Define the subspaces $\theta_n \subset C_n(\Gamma)$, $n > 0$, as those $f \in C_n(\Gamma)$ such that $V_{-i}f = 0$.

THEOREM 6. *Let all notation be as above, then*
1. $J\theta_n = \theta_n$,
2. $L(\lambda)\theta_n = \theta_n$, $\lambda \in \Gamma/(n) = \Lambda(n)/\Gamma(n)$.

The proof of Theorem 6 follows immediately from the fact that $JV_{-i} = -iV_{-i}$ and $L(\lambda)(X + iY) = X + iY$ because we are using left invariant vector fields.

In Auslander and Tolimieri it was shown that the following is true.

THEOREM 7. *Let all notation be as above. Then*
1. $\dim \theta_n = n$.
2. $\Sigma_{n>1}\theta_n \subset \mathbf{C}^\infty(\Gamma\backslash N)$ *is a subalgebra with no divisors of zero and* $\theta_n\theta_m \subset \theta_{n+m}$.
3. $L(\Gamma/(n))|\theta_n$ *is an irreducible representation.*

4. The existence of L_A. We are now in a position to prove the existence of an algebra L_A and this combined with the results in §1 will prove our main theorem.

To achieve this goal, we begin by selecting in N a decomposition of N similar to that defined for $\Gamma/(n)$. To be precise let $A' = (x, 0, 0)$, $A = (x, 0, t)$, $B' = (0, y, 0)$ and $B = (0, y, t)$. Then A is a maximal abelian subgroup of N and $N = B' \times A$. Further, B is also a maximal abelian subgroup of N and $JA = B$ and $JB = A$.

In $\Lambda(n)/\Gamma(n)$ we define $A(n)$ as the image of $\Lambda \cap \Gamma(n)$ and similarly define $B'(n)$, $B(n)$ and $A'(n)$. Hence we may identify θ_n with $L^2(B'(n)) = L^2(Z/n)$ by an isomorphism M_n. We fix these isomorphisms and hook them all together to obtain a vector space isomorphism

$$\oplus \sum_{n>1} L^2(Z/n) \overset{M}{\leftarrow} \oplus \sum_{n>1} \theta_n$$

such that $M \cdot J = \mathfrak{F}M$ by the uniqueness of intertwining operators for irreducible unitary representations and an elementary computation. But as remarked before $J(fg) = J(f)J(g)$, $f, g \in \oplus\Sigma_{n>1}\theta_n$. Thus we have proven

THEOREM 8. *If we use M to induce an algebra on L, we obtain one of the algebras L_A.*

This completes the proof of our Main Theorem.

It still remains to identify the algebra $M(\oplus\Sigma_{n>1}\theta_n)$ among the algebras L_α, $\alpha = 1, 2, 3$. To do this we recall the Weil-Brezin map that connects $C^\infty(R)$ and $C_1(\Gamma)$. For $F \in \mathcal{S}(R)$, the Schwarz space in $C^\infty(R)$, we define

$$W(f)(x, y, t) = e^{2\pi it}e^{\pi ixy}\sum_{l\in Z} f(y + l)e^{2\pi ilx} \in C_1(\Gamma).$$

Define $g_0 \in C_1(\Gamma)$ as $W(\exp - \pi y^2)$. Then one verifies as in Auslander and Tolimieri [2] that $\theta_1 = Cg_0$. We let

$$Y_1 = g_0, \quad Y_2^2 = L\left(\tfrac{1}{2}, \tfrac{1}{2}, 0\right)g_0^2,$$

and

$$Y_3^3 = L\left(\tfrac{1}{2}, 0, 0\right)Y_1L\left(0, \tfrac{1}{2}, 0\right)Y_1L\left(\tfrac{1}{2}, \tfrac{1}{2}, 0\right)Y_1.$$

By elementary computations, we have

$$JY_1 = Y_1, \quad JY_2^2 = (-1)Y_2^2, \quad \text{and} \quad JY_3^3 = iY_3^3.$$

It remains to compute a and b such that

$$Y_3^6 + aY_1^4Y_2^2 + bY_2^6 = 0.$$

Substituting and cancelling we are looking for a and b such that

$$\left[L\left(\tfrac{1}{2}, 0, 0\right)Y_1L\left(0, \tfrac{1}{2}, 0\right)Y_1\right]^4 + aY_1^4 + b\left[L\left[\tfrac{1}{2}, \tfrac{1}{2}, 0\right]Y_1\right]^4 = 0.$$

We now use that Y_1 vanishes only at $(\frac{1}{2}, \frac{1}{2}, t)$ in $\Gamma\setminus N$. Hence $(L(\frac{1}{2}, 0, 0)Y_1L(0, \frac{1}{2}, 0)Y_1)(\frac{1}{2}, \frac{1}{2}, 0)$ is not zero. But this implies that $b \neq 0$. Similarly

$$\left(L\left(\tfrac{1}{2}, 0, 0\right)Y_1L\left(0, \tfrac{1}{2}, 0\right)Y_1\right)(0, 0, 0) \quad \text{is not zero.}$$

This implies $a \neq 0$. Hence $\oplus\Sigma_{n>1}\theta_n$ is isomorphic to $\mathbb{C}[X_1, X_2^2, X_3^3]/(X_3^6 + X_2^2X_1^4 + X_2^6)$.

Since the Gauss sum $G(x) = \Sigma_{0<j<n}\exp 2\pi ij^2/n$ is related to the trace of $\mathcal{F}_n$, we have also

$$G(n) = \begin{cases} (i + 1)\sqrt{n}, & n \equiv 0 \bmod 4, \\ \sqrt{n}, & n \equiv 1 \bmod 4, \\ 0, & n \equiv 2 \bmod 4, \\ i\sqrt{n}, & n \equiv 3 \bmod 4. \end{cases}$$

References

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